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Two-pebbling and odd-two-pebbling are not equivalent

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Abstract

Let $G$ be a connected graph. A configuration of pebbles assigns a nonnegative integer number of pebbles to each vertex of $G$. A move consists of removing two pebbles from one vertex and placing one pebble on an adjacent vertex. A configuration is solvable if any vertex can get at least one pebble through a sequence of moves. The pebbling number of $G$, denoted $\pi(G)$, is the smallest integer such that any configuration of $\pi(G)$ pebbles on $G$ is solvable. A graph has the two-pebbling property if after placing more than $2\pi(G) - q$ pebbles on $G$, where $q$ is the number of vertices with pebbles, there is a sequence of moves so that at least two pebbles can be placed on any vertex. A graph has the odd-two-pebbling property if after placing more than $2\pi(G) - r$ pebbles on $G$, where $r$ is the number of vertices with an odd number of pebbles, there is a sequence of moves so that at least two pebbles can be placed on any vertex. In this paper, we prove that the two-pebbling and odd-two-pebbling properties are not equivalent.

Keywords: graph pebbling, Lemke graph, two-pebbling, odd-two-pebbling

1. Introduction

Let $G$ be a connected graph. A configuration assigns a nonnegative number of pebbles to the vertices of $G$. For a configuration $C$, we define $C(v)$ to be the number of pebbles on vertex $v$, and if $U$ is a subset of vertices of $G$, then $C(U)$ is the total number of pebbles on the vertices in $U$. A pebbling move (or just move) removes two pebbles from one vertex and places one pebble on an adjacent vertex. A vertex $v$ is reachable under some configuration if it is possible to move a pebble to $v$ through a sequence of pebbling moves. A configuration is solvable if all vertices are reachable. The pebbling number rooted at a vertex $v$ in $G$, $\pi(G, v)$, is defined as the smallest number of pebbles so that for any configuration of $\pi(G, v)$ pebbles, $v$ is reachable. The pebbling number of a graph is $\pi(G) = \max_{v \in V(G)} (\pi(G, v))$. 

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A graph $G$ has the two-\textit{pebbling property} if for every configuration of more than $2\pi(G) - q$ pebbles, where $q$ is the number of vertices with pebbles, it is possible to move 2 pebbles to any vertex. A \textit{violating configuration} for a vertex $v$ of $G$ is any configuration of more than $2\pi(G) - q$ pebbles such that two pebbles cannot be moved to $v$. A graph that does not have the two-\textit{pebbling property} is called a \textit{Lemke graph}.

The two-\textit{pebbling property} was introduced by Chung [1]. Most graphs have the two-\textit{pebbling property} [2]. In fact, only a handful of families of Lemke graphs have been found [3, 4, 5, 6, 7]. Graham’s Conjecture states for any two graphs $G$ and $H$, $\pi(G\square H) \leq \pi(G)\pi(H)$, where $G\square H$ is the Cartesian product of $G$ and $H$ [1]. Graham’s conjecture has been studied by numerous researchers, and many results that affirm the conjecture rely on the two-\textit{pebbling property} [1, 3, 5, 8, 9, 10].

A graph $G$ has the odd-two-\textit{pebbling property} if for every configuration of more than $2\pi(G) - r$ pebbles, where $r$ is the number of vertices with an odd number of pebbles, it is possible to move 2 pebbles to any vertex [5]. Note that any graph which has the two-\textit{pebbling property} also has the odd-two-\textit{pebbling property}. All Lemke graphs found to date also do not have the odd-two-\textit{pebbling property}. This is true even of more recent Lemke graphs [6, 7]. Wang conjectured that two-\textit{pebbling} and odd-two-\textit{pebbling} are equivalent [5]. We present a graph that has the odd-two-\textit{pebbling property} but does not have the two-\textit{pebbling property}, proving that the properties are not equivalent.

2. General Results

The following is a somewhat obvious but powerful tool in analyzing Lemke graphs.

**Theorem 1.** Let $C$ be a violating configuration on graph $G$ for root $r$ with $2\pi(G) - q + k$ pebbles, where $k \geq 1$. Then it is impossible to place a pebble on $r$ using less than $\pi(G) - q + k + 1$ pebbles.

**Proof.** If $\pi(G) - q + k$ pebbles are used to place one pebble on $r$, $\pi(G)$ pebbles are left on $G$ so a second pebble can be moved to $r$.

In our arguments related to the two-\textit{pebbling} property, we will often state that the root can be reached using $\pi(G) - q + 1$ pebbles and leave implicit the fact that a second pebble can be moved to the root by Theorem 1, implying that the given configuration is not a violating configuration for the given root.

**Lemma 2.** Let $G$ be a graph with $n$ vertices and let $C$ be a violating configuration for root $r$. Then $q < n$ and $C(r) = 0$.

**Proof.** If $q = n$, then there are at least $2\pi(G) - n + 1 \geq 2n - n + 1 = n + 1$ pebbles on $n$ vertices. Since every vertex has at least one pebble and at least one vertex has at least two pebbles, a second pebble can be moved to any vertex. Clearly $C(r) < 2$. If $C(r) = 1$, then there are at least $2\pi(G) - q + 1 - 1 =$
\[\pi(G) + (\pi(G) - q) \geq \pi(G)\] other pebbles on the graph and a second pebble can be moved to \(r\).

**Lemma 3.** Let \(G\) be a Hamiltonian graph with \(n\) vertices, \(C\) a configuration with \(p \geq n + 2\) pebbles on \(q = n - 1\) vertices. Then two pebbles can be moved to any vertex in \(G\).

**Proof.** Since some vertex has at least two pebbles, any vertex that already has a pebble can get a second pebble by pebbling along the Hamiltonian cycle. Let \(r\) be the vertex without a pebble. Since \(p = n + 2\), either two vertices, \(u\) and \(v\), have at least two pebbles or some vertex \(u\) has 4 pebbles. In the first case, a pebble can be moved to \(r\) from each of \(u\) and \(v\) along two disjoint paths that are part of the Hamiltonian cycle. Similarly, if some vertex has 4 pebbles, two pebbles can be moved to \(r\) from \(u\) by following two disjoint paths along the Hamiltonian cycle.

**Lemma 4.** Let \(C\) be a violating configuration, \(u\) be a vertex with \(C(u) \geq 3\), and assume \(C(v) = 0\) for some neighbor of \(v\) of \(u\). Create configuration \(C'\) from \(C\) by moving one pebble from \(u\) to \(v\). Then \(C'\) is a violating configuration.

**Proof.** Let \(C\) be a violating configuration for some root \(r\) with \(p\) pebbles on \(q - 1\) vertices such that \(C(u) \geq 3\), and let \(v\) be a neighbor of \(u\) with \(C(v) = 0\). Since \(C\) is a violating configuration, \(p + q - 1 > 2\pi(G)\). Then \(C'\) has \(p - 1\) pebbles on \(q\) vertices. Since \(p - 1 + q > 2\pi(G)\) and \(r\) is still not reachable with two pebbles, \(C'\) is clearly a violating configuration.

**Corollary 5.** Let \(G\) be a graph that has no violating configurations with pebbles on \(q\) vertices and let \(C\) be a violating configuration with pebbles on \(q - 1\) vertices. If \(C(u) \geq 3\), then for each neighbor \(v\) of \(u\), \(C(v) \geq 1\). Equivalently, if \(C(v) = 0\), then \(C(u) \leq 2\) for each neighbor \(u\) of \(v\).

The following lemma is straightforward.

**Lemma 6.** Let \(P_n\) be a path on \(n\) vertices, \(K_3\) be a clique on 3 vertices with vertex set \(V(K_3) = \{v_1, v_2, v_3\}\), and let \(C\) be a pebbling configuration.

1. If \(n \leq 4\) and \(C(P_n) \geq n + 1\), then at least two pebbles can be moved to one of its endpoints.
2. If \(C(K_3) \geq 4\), then it is possible to move 2 pebbles to at least two of its vertices.
3. If \(C(K_3) \geq 5\), then 2 pebbles can be moved to any of its vertices.
4. If \(C(K_3) \geq 6\), then 4 pebbles can be moved to one of its vertices. Further, if \(C(v_1) + C(v_2) \geq 6\) then 2 pebbles can be placed on both \(v_1\) and \(v_2\) simultaneously.
5. If \(C(K_3) = 7\) and 4 pebbles cannot be moved to \(v_1\) or \(v_2\), then \(C(v_3) = 5\) and \(C(v_1) = C(v_2) = 1\) or \(C(v_1) = 7\) and \(C(v_2) = C(v_3) = 0\).
6. If \(C(K_3) = 8\) and 4 pebbles cannot be moved to \(v_1\), then \(C(v_1) = 0\) and \(C(v_2)\) and \(C(v_3)\) are both odd.
7. If \(C(K_3) \geq 9\), then 4 pebbles can be moved to any of its vertices.
8. If \(C(K_3) \geq 14\) and each vertex has at least one pebble, then 4 pebbles can be moved to any two of its vertices simultaneously.
3. The new Lemke graph

When the algorithm from [6] to determine whether or not a graph has the two-pebbling property was run on all ten-vertex graphs with diameter three, several new Lemke graphs were discovered with a very interesting property: all of the violating configurations have at least one vertex with an even number of pebbles. In other words, these are Lemke graphs that have the odd-two-pebbling-property. Since this was an unexpected result, it seemed prudent to verify it. The goal of this paper is to prove that one of these graphs, $H$ (see Figure 1), does not have the two-pebbling property but does have the odd-two-pebbling-property, proving that these two properties are not equivalent. We will proceed by showing that $\pi(H) = 10$ and then prove that $H$ has exactly 6 violating configurations, none of which satisfy the conditions of the odd-two-pebbling property.

![Figure 1: The new Lemke graph, $H$](image)

Let $T_i$ be subgraph induced by vertices $\{a_i, b_i, c_i\}$ for $i \in \{1, 2, 3\}$. Let $C$ be a configuration on $H$. Let $p_i = C(T_i)$ and $q_i$ be the number of vertices on $T_i$ with pebbles. Finally, let $\alpha_i = C(a_i)$, $\beta_i = C(b_i)$, $\gamma_i = C(c_i)$, and $\delta = C(d)$.

If moves are made on a configuration $C$, the result is a new configuration that is usually given a new name (e.g. $C'$). To simplify the notation in proofs, we will often continue to call the configuration $C$ and use definitions from above even after moves have been made.

4. Pebbling Number

In this section we show that $\pi(H) = 10$. Clearly $\pi(H, v) \geq 10$ for all $v$. Due to the symmetry of $H$, we prove that $\pi(H, d) = \pi(H, a_3) = \pi(H, c_3) = 10$ and the result follows.

**Theorem 7.** $\pi(H, d) = 10.$
Proof. Let $C$ be a configuration with 10 pebbles such that $d$ is unreachable. By Lemma 6.3, $p_i \leq 4$ for $i \in \{1, 2, 3\}$. Without loss of generality, we may assume that $p_1 = 4$ and that $\alpha_1 = 3$ and $\beta_1 = 1$ (due to symmetry and unreachability of $d$). Further, it is impossible to move a pebble from $T_2$ or $T_3$ to $T_1$. If $p_2 = 4$ or $p_3 = 4$, Lemma 6.2 implies that a pebble can be moved to either $d$ or to $T_1$, so $p_2 = p_3 = 3$. The pebbles on $T_2$ do not allow a pebble to be moved to either $d$ or $b_1$. Thus, either $\alpha_2 = \beta_2 = \gamma_2 = 1$ or $\beta_2 = 3$ and $\alpha_2 = \gamma_2 = 0$. In the former case, $d$ is clearly reachable along the path $(a_1, b_1, a_2, c_2, d)$, so $\beta_2 = 3$ and $\alpha_2 = \gamma_2 = 0$. A similar argument shows that $\alpha_3 = 3$ and $\beta_3 = \gamma_3 = 0$. But then a pebble can be moved from $a_3$ to $b_2$ so that $b_2$ has four pebbles and $d$ is reachable. Therefore, $\pi(H, d) = 10$.

Theorem 8. $\pi(H, c_3) = 10$.

Proof. Let $C$ be a configuration of 10 pebbles on $H$. Without loss of generality, assume $p_1 \geq p_2$. If any of $\{b_3, a_3, d\}$ has two or more pebbles $c_3$ is reachable, so assume otherwise. There are 4 cases to consider.

Case 1: All three of $\{b_3, a_3, d\}$ have one pebble. Then $p_1 \geq 4$ and $c_3$ is reachable by Lemma 6.2.

Case 2: Two of $\{b_3, a_3, d\}$ have one pebble. If $p_1 \geq 5$, then both $b_3$ and $d$ are reachable from $T_1$ by Lemma 6.3. Since at least one of these has a pebble, $c_3$ is reachable. Otherwise, $p_1 = p_2 = 4$. If $\delta = 0$, then $\beta_3 = \alpha_3 = 1$ and $c_3$ is reachable unless $\alpha_1 = \beta_2 = 0$. In this case, a pebble can be moved to $d$ from both $T_1$ and $T_2$, so $c_3$ is reachable. If $\delta = 1$, then without loss of generality, $\alpha_3 = 0$ and $\beta_3 = 1$, and Lemma 6.2 implies that a pebble can be moved to either $d$ or $b_3$ from $T_1$, making $c_3$ reachable.

Case 3: There is one pebble on $\{b_3, a_3, d\}$. In this case, $p_1 \geq 5$. If $\beta_3 = 1$ or $\delta = 1$, $c_3$ is reachable by Lemma 6.3, so assume $\alpha_3 = 1$. This implies that $\beta_2 \leq 1$. If $p_1 \geq 8$, clearly $c_3$ is reachable from $T_1$. This leaves 3 subcases.

Case 3.1: $p_1 = 5$. Then by Lemma 6.3, a pebble can be moved from $T_1$ to $T_2$, putting 5 pebbles on $T_2$, which allows for a move to $a_3$, making $c_3$ reachable.

Case 3.2: $p_1 = 6$. Then $p_2 = 3$. If $\beta_2 = 1$, then $a_2$ either has a pebble or can receive one from $c_2$, so a pebble can be moved from $T_1$ along the path $(a_2, b_2, a_3, c_3)$. If $\beta_2 = 0$, there are two cases to consider. If $\alpha_2 = 3$, a move can be made from $T_1$ to $a_2$, making $c_3$ reachable through $a_3$. Otherwise, $\alpha_2 \leq 2$ and $\gamma_2 \geq 1$, so $d$ can be reached from both $T_1$ and $T_2$ and $c_3$ is reachable.

Case 3.3: $p_1 = 7$. Then $p_2 = 2$. By Lemma 6.5, if $c_3$ is not reachable from $T_1$, then either $\beta_1 = 5$ and $\alpha_1 = \gamma_1 = 1$, or $\beta_1 = 7$ and $\alpha_1 = \gamma_1 = 0$. If $d$, $b_1$, or $a_1$ is reachable from $T_2$, the configuration is solvable. Thus, two vertices in $T_2$ have one pebble. If $\beta_2 = 0$, then one pebble from $T_1$ can be moved through $T_2$ to $d$, leaving 5 pebbles on $T_1$, allowing another pebble to reach $d$. Otherwise, $\beta_2 = 1$ and we move two pebbles from $b_1$ to $a_2$ and then pebble along the path $(a_2, b_2, a_3, c_3)$.

Case 4: There are no pebbles on $\{b_3, a_3, d\}$. If $p_1 = p_2 = 5$, then 2 pebbles can be moved to $d$ and one to $c_3$. If $p_1 \geq 8$, clearly $c_3$ is reachable. This leaves two cases.
Case 4.1: $p_1 = 6$. Then $p_2 = 4$. If $d$ is reachable from $T_2$, then $c_3$ is reachable. Otherwise, either $\beta_2 = 3$ and $\alpha_2 = 1$ or $\beta_2 = 1$ and $\alpha_2 = 3$. If $\beta_2 = 3$, then move a pebble from $b_2$ to $a_3$. By Lemma 6.3, two pebbles can be moved to $b_1$ and then a pebble can be moved along the path $(b_1, a_2, b_2, a_3, c_3)$. If $\alpha_2 = 3$, a move from $a_2$ to $b_1$ would place $7$ pebbles on $T_1$. If that configuration is unsolvable for $c_3$, Lemma 6.5 implies that in the initial configuration either $\alpha_1 = \gamma_1 = 1$ and $\beta_1 = 4$, or $\beta_1 = 6$. In either case, moving a pebble from $b_1$ to $a_2$ instead allows pebbling to $d$ from both $T_1$ and $T_2$, making $c_3$ reachable.

Case 4.2: $p_1 = 7$. Then $p_2 = 3$. If $c_3$ is unreachable from $T_1$, then Lemma 6.5 implies that $\beta_1 = 5$ and $\alpha_1 = \gamma_1 = 1$, or $\beta_1 = 7$ and $\alpha_1 = \gamma_1 = 0$. If a pebble can be moved from $T_2$ to $T_1$, then $c_3$ is reachable since $T_1$ now has $8$ pebbles. If a pebble can be moved from $T_2$ to $d$, $c_3$ is also reachable. Otherwise, $\beta_2 = 3$ or $\alpha_2 = \beta_2 = \gamma_2 = 1$. If $\beta_2 = 3$, then $2$ pebbles can be moved from $b_1$ to $a_2$, one pebble from $b_2$ to $a_3$, and then one pebble can be moved along the path $(b_1, a_2, b_2, a_3, c_3)$. If $\alpha_2 = \beta_2 = \gamma_2 = 1$, move along the path $(b_1, a_2, c_2, d)$ and the remaining pebbles on $T_1$ allow a second pebble to be moved to $d$, so $c_3$ is reachable.

Let $H_1$ be the subgraph induced by the set of vertices $\{a_1, b_1, c_1, b_3\}$ and $H_2 = H \setminus H_1$. We will prove several results that will be used in the next theorem.

**Lemma 9.** Let $C$ be a configuration on $H$.

1. If $p_2 = 4$, then one pebble can be moved to $a_3$ unless $\beta_2 = 0$ and $\alpha_2$ and $\gamma_2$ are both odd.
2. If $C(H_2) = 6$, then a pebble can be moved to $a_3$ unless $\delta = \alpha_2 = 3$.
3. If $C(H_2) \geq 7$, then a pebble can be moved to $a_3$.

**Proof.** The proof of statement 1 is straightforward.

For statement 2, let $C(H_2) = 6$ and assume $\alpha_3 = 0$. By Lemma 6.3, $a_3$ is reachable if $p_2 \geq 5$. Thus, assume $p_1 \leq 4$ and therefore $\gamma_3 + \delta \geq 2$.

If $\gamma_3 + \delta = 2$, then $p_2 = 4$. By statement 1, we can assume $\alpha_2 = 1$ and $\gamma_2 = 3$ or $\alpha_2 = 3$ and $\gamma_2 = 1$. If $\gamma_3 = \delta = 1$, $a_3$ is clearly reachable. Otherwise, $\delta = 2$, and we can get $4$ pebbles to either $a_2$ or $c_2$, thus allowing a pebble to be moved to $a_3$.

If $\gamma_3 + \delta = 3$, then $\delta = 3$ and $\gamma_3 = 0$ or we can clearly reach $a_3$. In this case, $p_2 = 3$ and unless $\alpha_2 = 3$ (the exception in the statement), either $2$ pebbles can be moved to $b_2$ or one more pebble to $d$, and $a_3$ is reachable. Finally, $a_3$ is clearly reachable if $\gamma_3 + \delta \geq 4$.

For statement 3, if $C(H_2) = 7$, it is possible to remove one pebble from $C$ and avoid the configuration with $\delta = \alpha_2 = 3$. By statement 2, $a_3$ is reachable.

**Theorem 10.** $\pi(H, a_3) = 10$.

**Proof.** Let $C$ be a configuration of $10$ pebbles on $H$ and assume $\alpha_3 = 0$. By Lemma 9.3, $a_2$ is reachable if $C(H_2) \geq 7$. This leaves $6$ cases.

Case 1: $C(H_2) = 6$. By Lemma 9.2, $a_3$ is reachable unless $\delta = \alpha_2 = 3$. In this case, a pebble can be moved from $a_2$ to $b_1$ so that the path $\{b_3, a_1, b_1, c_1\}$
has 5 pebbles. By Lemma 6.1, two pebbles can be moved to either \(c_1\), in which case a fourth pebble can be added to \(d\), or to \(b_3\). In both cases, \(a_3\) can be reached.

Case 2: \(C(H_2) = 5\). Then \(C(H_1) = 5\) and by Lemma 6.1 at least 2 pebbles can be moved to \(b_3\) or \(c_1\) (by considering the path \(\{b_3, a_1, b_1, c_1\}\)), and at least 2 pebbles can be moved to \(b_3\) or \(b_1\) (by considering the path \(\{b_3, a_1, c_1, b_1\}\)). If two pebbles can be moved to \(b_3\), then \(a_3\) is reachable, so we can assume that 2 pebbles can be moved to either \(c_1\) or \(b_1\). If \(\alpha_2 = 3\), move a pebble from \(b_1\) to \(a_2\). Otherwise, move a pebble from \(c_1\) to \(d\). In either case, \(H_2\) now has 6 pebbles and \(\alpha_2 \neq 3\), so \(a_3\) is reachable by Lemma 9.2.

For the remaining cases, since \(C(H_1) \geq 6\), we can assume \(\beta_3 = 0\) since otherwise \(a_3\) is reachable. Thus, \(p_1 = C(H_1) \geq 6\). We will assume that \(a_3\) is not reachable from \(T_1\), so Lemma 6.4 implies that 4 pebbles can be moved to either \(b_1\) or \(c_1\). This implies that two pebbles can be moved to either \(d\) or \(\alpha_2\) from \(T_1\).

Case 3: \(C(H_2) = 4\). If \(\alpha_2 = 3\), move a pebble from \(T_1\) to \(\alpha_2\) and \(a_3\) is reachable. Similarly if \(\delta = 3\). Otherwise, move two pebbles to either \(d\) or \(\alpha_2\) from \(T_1\) so that \(C(H_2) = 6\). Since it is not that case that both \(\alpha_2 = 3\) and \(\delta = 3\), \(a_3\) is reachable by Lemma 9.2.

For the remaining cases \(C(H_2) \leq 3\), so \(p_1 \geq 7\). We will assume that 4 pebbles cannot be moved to \(a_1\) since otherwise \(a_3\) is reachable. Thus, \(\alpha_1 \leq 1\), so \(\beta_1 + \gamma_1 \geq 6\), and by Lemma 6.4, two pebbles can be placed on \(b_1\) and \(c_1\) simultaneously.

Case 4: \(C(H_2) = 3\), so \(p_1 = 7\).

Case 4.1: \(p_2 = 3\). If 4 pebbles can be moved to \(b_1\) from \(T_1\), then 2 pebbles can be moved from \(T_1\) to \(T_2\) and by Lemma 6.3, \(a_3\) is reachable. If 4 pebbles cannot be moved to either \(a_1\) or \(b_1\), then by Lemma 6.5, \(\gamma_1 \geq 5\), so a pebble can be added to either \(a_2\) or \(c_2\). Since either \(\beta_2 = 1\) or the parity of \(\alpha_2\) and \(\gamma_2\) differ, it is possible to move to either \(a_2\) or \(c_2\) so that \(a_3\) is reachable by Lemma 9.1.

Case 4.2: \(p_2 = 2\). Then \(\delta = 1\) or \(\gamma_3 = 1\). If 4 pebbles can be moved to \(b_1\) from \(T_1\), then by Lemma 9.1, we can assume \(\gamma_2 = \alpha_2 = 1\) and \(\beta_2 = 0\) since otherwise \(a_3\) is reachable. Move 2 pebbles to both \(b_1\) and \(c_1\). If \(\delta = 1\), move a pebble along the paths \((b_1, a_2, b_2)\) and \((c_1, d, c_2, b_2)\) so that \(b_2\) has two pebbles. If \(\gamma_3 = 1\), move a pebble from \(c_1\) to \(d\) and along the path \((b_1, a_2, c_2, d)\). In either case, \(a_3\) can be reached.

If 4 pebbles cannot be moved to either \(a_1\) or \(b_1\), by Lemma 6.5, either \(\gamma_1 = 5\) and \(\beta_1 = 1\) or \(\gamma_1 = 7\) and \(\beta_1 = 0\). Since 2 pebbles can be moved to \(d\), \(a_3\) is reachable if \(\gamma_3 = 1\), so assume \(\delta = 1\). If \(\gamma_1 = 7\) then we can move 3 more pebbles to \(d\). Otherwise, \(\gamma_1 = 5\) and \(\alpha_1 = \beta_1 = 1\). Since \(p_2 = 2\), there are four possibilities. If \(\beta_2 = 1\), then either \(\alpha_2 = 1\) or \(\gamma_2 = 1\) and a second pebble can be added to either \(a_2\) or \(c_2\) and then to \(b_2\). If \(\gamma_2 = 2\) or \(\alpha_2 = 2\) then 4 pebbles can be moved to \(a_1\). If \(\gamma_2 = \alpha_2 = 1\) then move a pebble along the paths \((c_1, b_1, a_2, b_2)\) and \((c_1, d, c_2, b_2)\) so \(b_2\) has two pebbles. In all cases, \(a_3\) is reachable.

Case 4.3: \(p_2 \leq 1\). Then \(\delta + \gamma_3 \geq 2\). If \(\gamma_3 \geq 2\), \(\delta \geq 3\), or both \(\delta \geq 1\) and \(\gamma_3 \geq 1\), then \(a_3\) is clearly reachable. Thus, \(\delta = 2\) and \(\gamma_3 = 0\). Then \(p_1 = 1\)
and either 4 pebbles can be moved to $c_1$ or, by Lemma 6.5, either $\beta_1 = 5$ and $\gamma_1 = \alpha_1 = 1$ or $\beta_1 = 7$ and $\alpha_1 = \gamma_1 = 0$. If $\beta_2 = 1$ or $\gamma_2 = 1$ then 2 pebbles can be moved to $b_2$, so $\alpha_2 = 1$. If $\beta_1 = 7$, we can move 3 more pebbles to $a_2$. Otherwise, $\beta_1 = 5$ and $\alpha_1 = \gamma_1 = 1$, so a second pebble can be moved to $a_1$ from $d$ and 2 more pebbles can be moved to $a_1$ from $b_1$. In any case, $a_3$ is reachable.

Case 5: $C(H_2) = 2$. Then $p_1 = 8$ and if $a_3$ is not reachable from $T_1$, then Lemma 6.6 implies that $\alpha_1 = 0$ and $\beta_1 + \gamma_1 = 8$, where both are odd. No matter how these pebbles are placed, both $d$ and $a_2$ are reachable with 2 pebbles from $T_1$, and one of them can receive 3 pebbles. Thus, $a_3$ can be reached if $b_2$ or $c_3$ has one pebble, $d$, $c_2$, or $a_2$ has two pebbles, or both $d$ and $a_2$ have one pebble. Thus, we can assume either $\delta = \gamma_2 = 1$ or $\gamma_2 = \alpha_2 = 1$, and it is straightforward to verify that $a_3$ can be reached from any of the eight configurations on $T_1$.

Case 6: $C(H_2) \leq 1$. Then $p_1 = 9$ and the $a_3$ is reachable by Lemma 6.7.

5. Two-Pebbling Property

Lemma 11. Let $C$ be a violating configuration on $H$ with pebbles on $q$ vertices. Then $4 \leq q \leq 7$.

Proof. Let $C$ be a configuration of $p = 21 - q$ pebbles on $q$ vertices of $H$. If $q = 1$, $p = 20 = 2 \pi(H)$ and 2 pebbles can be moved to any vertex.

If $q = 2$, $p = 19$, and some vertex $u$ has at least ten pebbles. Since the diameter of $H$ is 3, moving from $u$ to any other vertex uses at most 8 pebbles, leaving at least 11 pebbles, enough to move a second pebble that vertex.

If $q = 3$, $p = 18$. Each of the three vertices with a pebble has at most 7 pebbles since otherwise one pebble can be placed on any other vertex leaving $\pi(H)$ pebbles on the graph, so a second pebble can be moved to that vertex. Thus each of the three vertices with pebbles, $u$, $v$, and $w$, has between 4 and 7 pebbles. No matter which vertices $u$, $v$, and $w$ are, every vertex is within distance two of one of them. Thus, one pebble can be moved to any root using 4 pebbles, leaving 14 pebbles to move a second pebble.

By Lemma 2, $q \leq 9$ and $r$ has no pebbles. If $q = 9$, $p = 12$ and since $H$ is Hamiltonian, the result follows from Lemma 3.

If $q = 8$, $r$ and some other vertex $v$ have no pebbles. Since $p = 13$ and $H \setminus \{v\}$ is Hamiltonian, the result follows from Lemma 3.

Theorem 12. There are no configurations on $H$ that violate the two-pebbling-property with $d$ as the root.

Proof. Let $C$ be a violating configuration for vertex $d$ with $21 - q$ pebbles on $q$ vertices. By Lemma 11, we only need to consider $4 \leq q \leq 7$. In all of these cases, $p \geq 14$. Therefore, $p_i \geq 5$ for some $i$. No matter how those pebbles are placed on $T_i$, $d$ is reachable using only 4 pebbles, and the result follows from Theorem 1.

Lemma 13. Let $C$ be a configuration on $H$ with $\alpha_1 \geq 1$. 


1. If $\alpha_1 + \gamma_1 \geq 7$, $\alpha_1 + \beta_1 \geq 7$, or $\alpha_1 + \beta_1 + \gamma_1 \geq 8$, then a pebble can be moved to $a_3$.

2. If $p_1 = 14$ and either $\beta_1 \geq 6$ and $a_2 \geq 1$ or $\gamma_1 \geq 6$ and $d \geq 1$, two pebbles can be moved to $a_3$.

**Proof.** The proof of statement 1 is straightforward. For the second statement, move 3 pebbles from $b_1$ to $a_2$ (or from $c_1$ to $d$) and then a pebble can be moved from $a_2$ (or $d$) to $a_3$. Since $p_1 = 8$ now, the result follows from statement 1.

**Lemma 14.** Let $C$ be a violating configuration on $H$ with root $a_3$ with pebbles on $q \leq 7$ vertices such that there are no violating configurations on $q+1$ vertices. Then $p_2 \leq 3$.

**Proof.** If $p_1 \geq 5$, Lemma 6.3 implies that $a_3$ is reachable from $T_2$. It is not too difficult to see that it requires at most 4 of the 5 pebbles, leaving at least 10 pebbles on $H$, allowing a second pebble to reach $a_3$. When $p_2 = 4$, each configuration either allows $a_3$ to be reachable with at most 4 pebbles or violates Corollary 5.

**Theorem 15.** $H$ has no violating configurations with root $a_3$.

**Proof.** Let $C$ be a violating configuration with $21 - q$ pebbles on $q$ vertices. By Lemma 11, we only need to consider $4 \leq q \leq 7$. In all of these cases, $p \geq 14$. By Lemma 14, $p_2 \leq 3$. Also, $\gamma_3 + \delta \leq 3$ and $\beta_3 \leq 1$ by Theorem 1. This implies that $p_1 \geq 7$. Corollary 5 implies that $q_1 = 3$. Using Theorem 1 again, $\beta_3 = 0$, and thus $p_1 \geq 8$. Once again, Corollary 5 implies that $1 \leq \alpha_1 \leq 2$ (a fact we use often when applying Lemma 13.1), so $\beta_1 + \gamma_1 \geq 6$. By Theorem 1, if $\beta_1 \geq 2$, at least one of $\alpha_2$ and $\beta_2$ is zero, and if $\gamma_1 \geq 2$, at least one of $\delta$ and $\gamma_3$ is zero. Since $\beta_1 + \gamma_1 \geq 6$, it follows that at least one of $\alpha_2$, $\beta_2$, $\delta$, and $\gamma_3$ is zero.

Case 1: $q = 7$. Then $p = 14$. Since $\alpha_3 = \beta_3 = 0$, exactly one other vertex has no pebbles. Thus, either $\delta = \gamma_3 = 1$ or $\alpha_2 = \beta_2 = 1$. In either case, $\gamma_2 = 1$.

Case 1.1: $\delta = \gamma_3 = 1$. Then $\gamma_1 = 1$, so $\beta_1 \geq 5$, and Corollary 5 implies that $1 \leq \alpha_2 \leq 2$, so that $\beta_2 = 0$. If $\alpha_2 = 2$, move a pebble along $(a_2, c_2, a_3)$ and Lemma 6.7 implies that $a_3$ is reachable with a second pebble since $p_1 = 9$. If $\alpha_2 = 1$, then $\alpha_1 + \beta_1 = 9$. Move along $(b_1, c_1, d, c_3, a_3)$ leaving $\alpha_1 + \beta_1 \geq 7$, so Lemma 13.1 applies.

Case 1.2: $\alpha_2 = \beta_2 = 1$. Then $\beta_1 = 1$, and $\delta + \gamma_3 = 1$, so $\alpha_1 + \gamma_1 = 9$. Pebble along $(c_1, b_1, a_2, b_2, a_3)$ leaving $\alpha_1 + \gamma_1 \geq 7$, so Lemma 13.1 applies.

Case 2: $q = 6$. Then $p = 15$. Since $p_2 \leq 3$ and $\delta + \gamma_3 \leq 3$, then $p_1 \geq 9$. Theorem 1 implies that either $\delta = 0$ or $\gamma_3 = 0$, and either $\alpha_2 = 0$ or $\beta_2 = 0$, and all other vertices have at least one pebble. This gives us four cases.

Case 2.1: $\delta = \alpha_2 = 0$. Corollary 5 implies that $a_1$, $b_1$, and $c_1$ each have at most two pebbles, contradicting the fact that $p_1 \geq 9$.

Case 2.2: $\delta = \beta_2 = 0$. Then $\gamma_3 = 1$, $\gamma_1 \leq 2$, and $\gamma_2 + \alpha_2 \leq 3$, so $p_1 \geq 11$ and $\beta_1 \geq 7$. If $\gamma_1 = 2$, move a pebble from $c_1$ to $d$ and then move a pebble along $(b_1, a_2, c_2, d, c_3, a_3)$, leaving $\alpha_1 + \beta_1 \geq 7$. If $\gamma_1 = 1$, there are two cases to consider. If $\alpha_2 + \gamma_2 = 3$, move a pebble from $T_2$ to $d$. Then move a pebble along
Lemma 16. Let $C$ be a configuration on $H$. Then for $i \in \{1, 2, 3\}$, the following hold.

1. If $p_i \geq 8$, or $p_i = 7$ and $q_i = 2$, then a pebble can be moved to $c_3$.
2. If $\delta = 1$ and $p_i + q_i \geq 13$, two pebbles can be moved to $c_3$.
3. If $p_i + q_i \geq 17$, two pebbles can be moved to $c_3$.

Proof. The statements are obvious when $i = 3$. Statement 1 follows from Lemma 6.5. To prove statement 2, when $q_i = 1$, use at most 4 pebbles from $T_i$ to move to $c_3$, leaving 8 on some vertex of distance at most 3 from $c_3$. For $q_i = 2$ and $q_i = 3$, make moves from $T_i$ to $d$ to $c_3$, and apply statement 1. For statement 3, apply statement 1 for $q_i = 1, 2$ and use Lemma 6.8 for $q_i = 3$.

Theorem 17. $H$ has exactly two violating configurations with $c_3$ as the root.
Proof. Let $C$ be a violating configuration with $21 - q$ pebbles on $q$ vertices. By Lemma 11, we only need to consider $4 \leq q \leq 7$. In all of these cases, $p \geq 14$. By Theorem 1, $c_3$ has no pebbles, $a_3$, $b_3$, and $d$ each have at most one pebble, and $a_1$, $c_1$, $c_2$, and $b_2$ each have at most three pebbles. We can assume that $p_1 \geq p_2$. Since $p_1 + p_2 \geq 11$, $p_1 \geq 6$. If both $a_1$ and $b_3$ have at least one pebble, a pebble can be moved to $c_3$ using four pebbles. Thus, at least one of $a_1$ or $b_3$ has no pebbles. Similarly, at least one of $c_1$ or $d$ has no pebbles.

Case 1: $q = 7$. Then $p = 14$, and since two of $a_1$, $b_3$, $c_1$, and $d$ have no pebbles, each of $b_1$, $a_2$, $b_2$, $c_2$, and $a_3$ has at least one pebble. Theorem 1 implies that $\alpha_2 = \beta_2 = \gamma_2 = \alpha_3 = 1$. Clearly $\alpha_1 + \beta_3 \leq 3$ and $\gamma_1 + \delta \leq 3$, so $\beta_1 \geq 4$. Pebble along the path $(a_1, a_2, b_2, a_3, c_3)$, leaving 8 pebbles on $\{a_1, b_1, c_1, b_3, d\}$. Since the subgraph induced by $\{a_1, b_1, c_1, b_3, d\}$ has $C_6$ as a spanning subgraph, and $\pi(C_6) = 8$, a second pebble can be moved to $c_3$.

For the remainder of the cases, $q \leq 6$ and $p \geq 15$. Since $p_1 \geq 6$, $\delta = \beta_3 = 0$ by Theorem 1. This and the fact that $\alpha_3 \leq 1$ implies that $p_1 + p_2 \geq 14$, so $p_1 \geq 7$. Corollary 5 implies that $1 \leq \alpha_1 \leq 2$ and $1 \leq \gamma_1 \leq 2$, so $\beta_1 \geq 3, \gamma_1 = 3$, and $\alpha_2 \geq 1$. By Theorem 1, at least one of $b_2$ or $a_3$ has no pebbles.

Case 2: $q = 6$. Then $p = 15$, exactly one of $b_2$ or $a_3$ has no pebbles, and $\alpha_2 = 1$ by Theorem 1.

Case 2.1: $\beta_2 = 0$. Then $1 \leq \alpha_2 \leq 2$ by Corollary 5 and $p_1 \geq 11$. If $\alpha_2 = 2$, move a pebble along the path $(a_2, c_2, d)$. If $\alpha_2 = 1$, then $p_1 = 12$ and we move a pebble along the path $(b_1, a_2, c_2, d)$. In both cases, $p_1 \geq 10$ and $\delta = 1$ after the moves, so Lemma 16.2 applies.

Case 2.2: $\alpha_3 = 0$. Since $q_2 = 3$ and $p_1 \geq p_2, 3 \leq p_2 \leq 7$. Since $\alpha_3 = \beta_3 = \delta = 0$, Corollary 5 implies that each of $\alpha_1, \gamma_1$, and $\beta_2$ is 1 or 2.

Case 2.2.1: $3 \leq p_2 \leq 6$. If $p_2 = 3$, then $a_2 = 1$, and $p_1 = 12$. Move a pebble along the path $(b_1, a_2, c_2, d)$. If $4 \leq p_2 \leq 5$, then $p_1 \geq 10$ and since $q_2 = 3$, a pebble can be moved from $T_2$ to $d$. If $p_2 = 6$, $p_1 = 9$, and since $q_2 = 3$, a pebble can be moved to both $d$ and $b_1$ from $T_2$. In all three cases, $p_1 \geq 10, q_1 = 3$, and $\delta = 1$ after the moves, so Lemma 16.2 applies.

Case 2.2.2: $p_2 = 7$. Then $p_1 = 8$ and $c_3$ can be reached from $T_1$ by Lemma 16.1. If $\beta_2 = 2$, $c_3$ can also be reached from $T_2$. Thus $\gamma_2 = \beta_2 = 1$ and $\alpha_2 = 5$. If $\gamma_1 = 2$, then $c_3$ can be reached using 5 pebbles from $c_1, a_2$, and $c_2$, so Theorem 1 applies. Thus $\gamma_1 = 1$. If $\alpha_1 = 2$, then $\beta_1 = 5$. Move two pebbles from $b_1$ to $a_1$ and then through $b_2$ to $c_3$. Then move along the paths $(a_2, b_1, c_1, d), (a_2, c_2, d)$, and from $d$ to $c_3$ with a second pebble. This implies that $\alpha_1 = \gamma_1 = \gamma_2 = \beta_2 = 1, \beta_1 = 6, and \alpha_2 = 5$. Because of the symmetry of the graph, if we remove our assumption that $p_1 \geq p_2, \beta_1 = 5$ and $\alpha_2 = 6$ also leads to a violating configuration. It is easy to see that if we add a pebble to either $b_1$ or $a_2$, it is possible to move two pebbles to $c_3$. Thus, when $q = 6$ there are exactly two violating configurations with $p = 15$, and none with $p \geq 16$.

Case 3: $q = 5$. Then $p = 16$ and $p_1 \geq 8$, so Theorem 1 implies $\gamma_3 = \delta = \beta_3 = 0, \alpha_1 = \gamma_1 = 1, and \gamma_2 = 2$. Exactly one of $a_3, b_2$, and $c_2$ has any pebbles.

Notice that there are exactly 16 configurations of pebbles with $q = 5$ and $p = 16$ that yield one of the violating configurations above after a move is made (8 for each), and it is easy to check that they are not violating configurations.
before the move. For instance, if \( \beta_1 = 8, \alpha_2 = 5, \) and \( \alpha_1 = \gamma_2 = \beta_2 = 1, \) move from \( b_1 \) to \( a_2 \) and apply Lemma 16.1 to both \( T_1 \) and \( T_2. \) Similarly for \( \beta_1 = 6, \alpha_2 = 5, \alpha_1 = 3, \) and \( \gamma_2 = \beta_2 = 1. \) Lemma 4 implies that we can assume for the remainder of the cases that if \( C(v) = 0, \) then \( C(u) \leq 2 \) for any neighbor \( u \) of \( v. \) Thus, \( 1 \leq \alpha_2 \leq 2 \) since at least one of its neighbors has no pebbles. Similarly, \( \beta_2 \leq 2. \)

If \( \beta_2 = 2, \) then \( \gamma_2 = \alpha_3 = 0 \) and \( \beta_1 \geq 10. \) If \( \alpha_2 = 2, \) move from both \( a_2 \) and \( b_2 \) to \( c_2 \) and then to \( d. \) If \( \alpha_2 = 1, \) then \( \beta_1 = 11. \) Pebble along the path \( (b_1, a_2, c_2) \) and then \( (b_2, c_2, d) \) leaving \( \beta_1 = 9. \) In either case, Lemma 16.2 applies.

If \( \beta_2 \leq 1, \) then \( \beta_2 + \gamma_2 + \alpha_3 = 1. \) Thus, either \( \beta_1 = 12 \) and \( \alpha_2 = 1, \) or \( \beta_1 = 11 \) and \( \alpha_2 = 2 \) and we move one pebble from \( a_2 \) to \( b_1. \) In both cases, Lemma 16.3 applies.

Case 4: \( q = 4. \) Then \( p = 17. \) Corollary 5 and Theorem 1 imply that \( \alpha_1 = \gamma_1 = 1, \) \( 1 \leq \alpha_2 \leq 2, \) and \( 13 \leq \beta_1 \leq 14, \) and Lemma 16.3 applies.

Given the symmetry of the graph, the following result is obvious.

**Theorem 18.** \( H \) has exactly 6 violating configurations.

**Theorem 19.** \( H \) does not have the two-pebbling property, but does have the odd-two-pebbling property.

**Proof.** \( H \) does not have the two-pebbling property by Theorem 18. All of the violating configurations have \( p = 15 \) and \( r = 5, \) and since \( 15 \neq 20 - 5 = 2\pi(H) - r, \) they do not violate the odd-two-pebbling property.


